Controlling transitions in a Duffing oscillator by sweeping parameters in time

Oleg Kogan*
Caltech, Pasadena, California 91125, USA

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We consider a high-Q Duffing oscillator in a weakly nonlinear regime with the driving frequency σ varying in time between σ_i and σ_f at a characteristic rate r. We found that the frequency sweep can cause controlled transitions between two stable states of the system. Moreover, these transitions are accomplished via a transient that lingers for a long time around the third, unstable fixed point of saddle type. We propose a simple explanation for this phenomenon, and find the transient lifetime to scale as $-(\ln|r-r_c|)/\lambda_r$, where r_c is the critical rate necessary to induce a transition and λ_r is the repulsive eigenvalue of the saddle. Experimental implications are mentioned.

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Mechanical systems such as nano electro-mechanical systems (NEMS) and micro electro-mechanical systems (MEMS) are often modeled as simple nonlinear driven oscillators. It is well known that even the simplest of nonlinear dynamical systems, such as those described by second-order ordinary differential equations, are notorious for exhibiting rich phenomenology [1]. One of the features of such systems is multistability, which strictly exists when all parameters are time independent. The subject of dynamic bifurcations [2–6] addresses some of the phenomenology that results from the time dependence of parameters in multistable systems. In this paper, we describe an overlooked phenomenon which occurs upon rapid sweeping of parameters in a damped, driven Duffing oscillator (which serves as an excellent model for NEMS [9,11]). In the case of adiabatic sweeping, a system initially situated at one of the quasi-fixed-points will remain close to it. When the rate of sweeping is significantly faster than the time scale determined by eigenvalues around the (instantaneous) fixed points (FPs), transitions occur such that after the rapid variation is finished, the system finds itself at a FP different from the one at which it was initially situated.

Let us consider an oscillator for which the position $q(t)$ evolves according to the following model: $\ddot{q}+(\omega_0/Q)\dot{q}+\omega_0^2 q(1+\alpha q^2)=2f\cos\phi(t)$. Here the instantaneous driving frequency is $\dot{\phi}$, ω_0 is the angular frequency of undamped infinitesimal vibrations, Q is a quality factor, α is a nonlinear coefficient, and f is the driving amplitude. We assume that $Q\gg 1$; a typical Q for NEMS for example, is in the range 10^3-10^4. Then $1/Q\equiv\epsilon$ serves as a natural measure of smallness. In addition to ϵ being small, in NEMS and MEMS applications, an important regime is for the nonlinear force and driving amplitude to be much smaller than the linear restoring force. Then letting $f=(\omega_0^2\epsilon^{3/2}/\alpha^{1/2})F$ and $q=(\epsilon^{1/2}/\alpha^{1/2})x$, as well as nondimensionalizing the time $t'=\omega_0 t$ (subsequently dropping the prime), we obtain

$$\ddot{x}+x=\epsilon[2F\cos\phi(t)-\dot{x}-x^3] \tag{1}$$

where $\dot{\phi}=1+\epsilon\sigma(t)$. For the case of time-independent F and σ, the amplitude versus σ response curve has the well-known frequency-pulled form [7,8]. For a given $F>F_{cr}=2^{3/2}/3^{5/4}$, there is a region of trivaluedness in oscillator amplitude vs σ with two stable branches and one unstable (middle) branch (Fig. 1). Such response curves have recently been measured for NEMS [9], indicating their Duffing-like behavior. For a given $\sigma>\sigma_{cr}=\sqrt{3/4}$, there is also a trivalued response function of amplitude vs F (Fig. 1).

We would like to explore what happens when the driving frequency, starting from the single-valued region (at σ_i) is rapidly swept into the trivalued region (ending at σ_f) at a constant F (a word about sweeping F at a constant σ will also be mentioned below). To illustrate and explain the phenomenology we will be using the piecewise linear ramp as shown in Fig. 2, although the phenomenology does not change qualitatively when smooth ramps are used. This work will probe the regime of $F/F_{cr}\sim O(1)$ during the ramp of σ, for instance, $F=2F_{cr}$. When $F\sim F_{cr}$, the size of the hysteresis $(\sigma_2-\sigma_1)$ is also of order 1 and the perturbation method is especially simple. The more complicated case of $F\gg F_{cr}$ [but small enough that the right-hand side (RHS) of Eq. (1) is still a perturbation over the harmonic oscillator equation] may be considered in a future work.

We discover that upon sweeping σ from lower values into the hysteresis, depending on the other parameters, the solution to Eq. (1) may jump onto the lower branch, as shown in an example in Fig. 3. The transitions have a peculiar feature of having a transient with lifetime τ much longer than the slow time scale of Eq. (1)—the damping time scale of order ϵ, often referred to as the "run-down time." The lifetimes τ can be defined as the width of the transient region (see Fig. 3) at some chosen fraction of its maximum. Now, there are four relevant "control knobs:" F, σ_i, σ_f, and $r\equiv 1$/(sweeping time). Consider a set of imaginary experiments, each per-

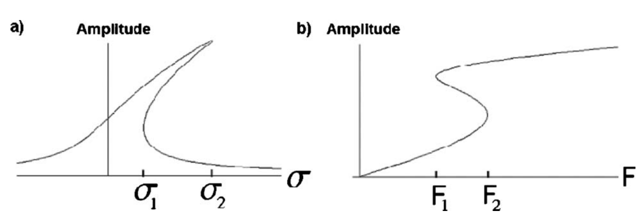

FIG. 1. Amplitude response: (a) constant F; (b) constant σ.

*oleg@caltech.edu

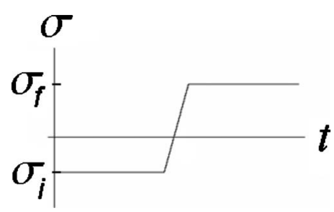

FIG. 2. Shape of the ramping function $\sigma(t)$. The quantity $\Delta\sigma$ is defined as $\sigma_f-\sigma_i$.

formed at a different sweep rate r, with three other parameters fixed. Depending on the value of the other parameters there may be a critical sweep rate r_{cr}, such that when r surpasses this r_{cr}, transitions will be induced. Moreover for r approaching very close to r_{cr}, τ will grow (see example in Fig. 5) and also the amplitudes and phases of these long transients tend to approach that of the middle (unstable) branch of the static Duffing oscillator (see example in Fig. 4). We learn that for r close to r_{cr} the solution moves onto the unstable branch, lives there for a time period τ, and then either moves onto the top branch if $r<r_{cr}$ or performs the transition onto the bottom branch if $r>r_{cr}$. The jump onto either the top or the bottom branch takes place long after reaching the static conditions (see Figs. 3 and 4). A numerical experiment performed at particular parameter values demonstrates a typical situation: in Fig. 5 we plotted on a semilogarithmic plot the lifetime τ versus $|r-r_c|$—it nearly follows a straight line. Thus, we learn that $\tau\propto-\ln|r-r_c|$.

Another numerical experiment, depicted in Fig. 6, was performed to measure r_c versus σ_f for $F\approx2.8F_{cr}$ and $\sigma_i=-2$. Two interesting features are immediately observed: (i) the critical rate necessary to induce a transition is not simply determined by the small parameter ϵ in Eq. (1); (ii) for $\sigma_f<\sigma_f^J$ transitions cannot be induced for *any* r. For σ_f slightly above σ_f^J, r_c behaves as $(\sigma_f-\sigma_f^J)^{-1/2}$. It is somewhat surprising to see singularities in this rather simple problem. Increasing σ_i moves σ_f^J to larger values and varying F does not significantly affect the shape of the $r(\sigma_f)$ curve. Transitions may occur by sweeping σ down; also by sweeping F and holding σ constant. The transition phenomenon persists for F large enough that the hysteresis is large (see comments in the last paragraph).

The regime of $F/F_{cr}\sim O(1)$ guarantees that $(\sigma_2-\sigma_1)\sim O(1)$; thus if σ_i is chosen sufficiently close to the hysteretic region, the jump in frequency during the sweep, $\sigma_f-\sigma_i\equiv\Delta\sigma$ is also $\sim O(1)$. This paves the way for a simple perturbation method. For example, we write the "multiple-scales" perturbative solution (see [8], for example) to Eq. (2) as $x=x_0(t,T,\ldots)+\epsilon x_1(t,T,\ldots)+O(\epsilon^2)$ where the slow time scale $T=\epsilon t$, and in general $T^{(n)}=\epsilon^n t$. Plugging this into the equation and collecting terms of appropriate orders of ϵ teaches us that $x_0=A(T)e^{it}+\text{c.c.}$, i.e., the solution is essentially a slowly modulated harmonic oscillator, with the modulation function satisfying the following amplitude equation (AE):

$$Fe^{i\delta(T)}-2i\frac{\partial A}{\partial T}-iA-3|A|^2A=0$$

where $d\delta/dT=\sigma(T)$. This AE holds for any sweep rate as long as $\Delta\sigma\sim O(1)$. Let $A=a(T)e^{\delta(T)}$. Then if we let $X=\text{Re}[a]$, and $Y=\text{Im}[a]$, we obtain

$$\frac{dX}{dT}=-\frac{1}{2}X+\sigma(T)Y-\frac{3}{2}(X^2+Y^2)Y, \tag{2}$$

$$\frac{dY}{dT}=-\sigma(T)X-\frac{1}{2}Y+\frac{3}{2}(X^2+Y^2)X-\frac{F}{2}. \tag{3}$$

These AEs are well known and appear in similar forms in literature [8,10]. For a certain range of parameters these equations give rise to a two-basin dynamics with a stable

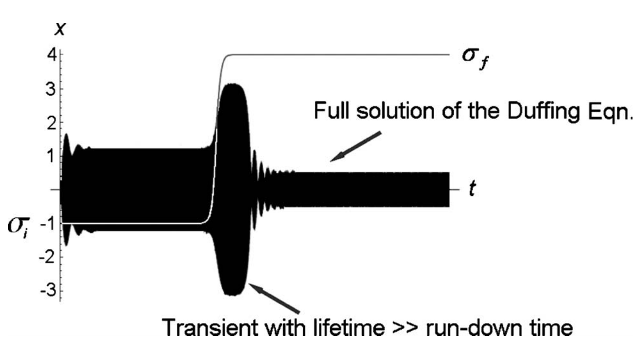

FIG. 3. Example of a transition with a long transient.

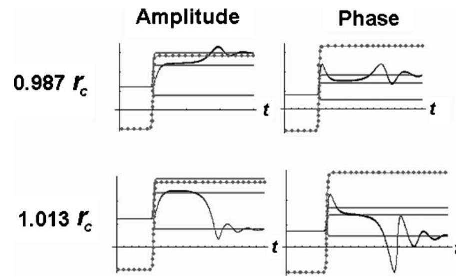

FIG. 4. Solutions kiss the unstable branch—envelope of the full numerical solution of Eq. (1) along with quasi-fixed-points [solutions to Eqs. (2) and (3) below with LHS set to 0] are shown. The function $\sigma(t)$ is displayed as a dotted curve.

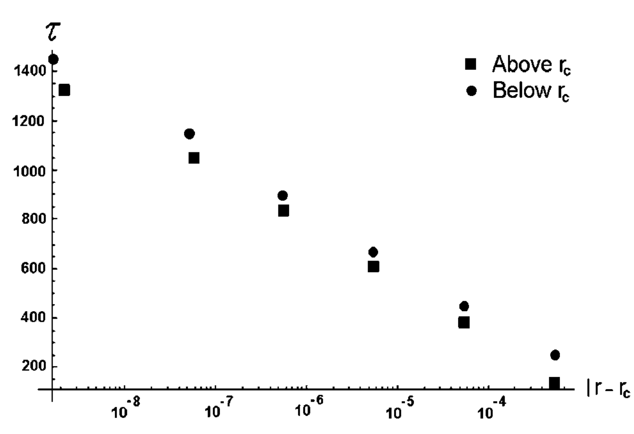

FIG. 5. Transient lifetime versus $|r-r_{cr}|$ for $F\approx2.8F_{cr}$, $\sigma_i=-2$, $\sigma_f=0.5(\sigma_1+\sigma_2)$.

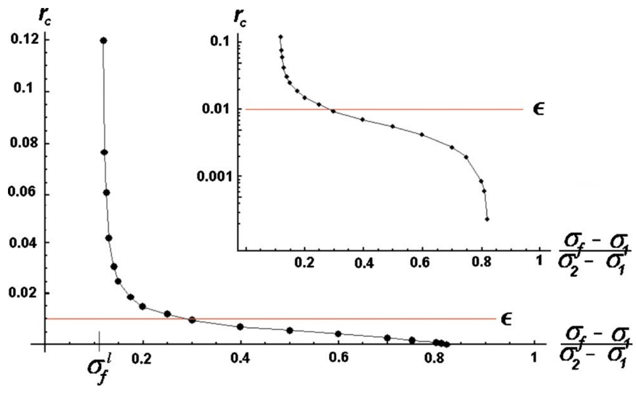

FIG. 6. (Color online) $F=2.8F_{cr}$, $\sigma_i=-2$. There is a singularity at σ_f^l. The size of the hysteresis was calculated with the first-order (in ϵ) theory and therefore it is an overestimation—the point where r_c goes to zero is the true bifurcation value of σ_2. The existence of singularity at σ_f^l is a genuine effect, since the first-order theory predicts the lower end of the hysteresis more accurately.

fixed point inside each basin. The basins are divided by a separatrix which happens to be the stable manifold of the unstable FP of saddle type (Fig. 7). Such basins have recently been mapped for a certain type of NEMS [11]. Next, consider a set of thought experiments, each sweeping σ at a different sweep rate r. Immediately after the sweep, the system will find itself somewhere in the two-basin space corresponding to conditions at σ_f, call it a point $(X_f(r),Y_f(r))$. If $(X_f(r),Y_f(r))$ lies in the basin that has evolved from the basin in which the system began before the sweep, then there will be no transition. If $(X_f(r),Y_f(r))$ lies in the opposite basin, then that corresponds to a transition. For very low r, during the sweep the system will follow closely the quasi-FP and there will be no transition. In the opposite extreme, infinite r, at the end of the sweep the system will not have moved at all, due to continuity of a dynamical system. This end point of $\{(X_f(\infty),Y_f(\infty))\}$ may lie in either basin depending on σ_f (Fig. 7).

In the numerical experiments that we considered, the first point of the $\{(X_f(r),Y_f(r))\}$ curve to cross the separatrix happens to be this tail[1] at $r=\infty$. This explains the $(\sigma_f-\sigma_f^l)^{-1/2}$ singularity mentioned earlier[2] (see Fig. 6). We note in passing that the theoretical reasoning outlined in this section is quite general and therefore explains transitions which occur under sweeping of F at fixed σ.

The point $(X_f(r),Y_f(r))$ serves as an initial condition for subsequent evolution at fixed σ_f. The $(X_f(r),Y_f(r))$ close to the separatrix (i.e., for r close to r_c) will flow toward the saddle and linger around it for a while. Because this lingering will take place close to the saddle, the linearized dynamics around the saddle should be a good approximation: $\mathbf{r}(T)=\delta l\mathbf{v}_r e^{\lambda_r T}+R\mathbf{v}_a e^{\lambda_a T}$, where $\mathbf{v}_r$ and λ_r are the repulsive eigenvector and eigenvalue, respectively, $\mathbf{v}_a$ and λ_a are the attractive eigenvector and eigenvalue, respectively, δl is the distance of $(X_f(r),Y_f(r))$ away from the separatrix along $\mathbf{v}_r$, and R is the characteristic radius of linearization around the saddle. The times at which the system crosses this circular boundary satisfy $R=\sqrt{\mathbf{r}(T)\cdot\mathbf{r}(T)}$. The first time, T_{in}, is negligible. The second time is $T_{out}\approx(1/\lambda_r)\ln(R/\delta l)$ (neglecting effects of $\mathbf{v}_a$). The lingering time is $\tau=T_{out}-T_{in}\approx T_{out}$. So

$$\tau=-\frac{1}{\lambda_r}\ln\left(\frac{\delta l}{R}\right)=-\frac{1}{\lambda_r}\ln\left(\frac{(dl/dr)|r-r_c|}{R}\right). \quad (4)$$

Here $l(r)$ is the distance along the curve $\{X_f(r),Y_f(r)\}$ from the place that it crosses the separatrix at $r=r_c$ to some point parametrized by r, so $\delta l=(dl/dr)|r-r_c|$. Thus we capture the logarithmic dependence of the transient time versus $|r-r_c|$.

We found that calculating the eigenvalues λ_r using the second-order theory as described in [10] *systematically* lowers the discrepancy in the slope of τ vs $|r-r_{cr}|$ computed from the exact Duffing equation and theoretical predictions (see Fig. 8), which leads us to conclude that the errors are due to inexactness of the first order AE, not due to incorrectness of the explanation of the cause of the transition phenomenon.

One can propose to use the described phenomenon to position Duffing-like systems onto the unstable (middle) branch (see Fig. 1)—the desire to do this has been expressed by workers in the NEMS community [12]. The first question is whether a necessary r_c is attainable. We see from Fig. 6 and the related discussion that for a vast range of parameters r_c is less than 0.1. Recall that in this paper r is defined simply as $1/(\text{sweep time})=1/\Delta T$. An experimentally relevant quantity is $\Delta\sigma/\Delta T$. In the present paper we are concerned with hysteresis widths of ~ 10 or less. Hence the $\Delta\sigma/\Delta T$ needed to cause a transition is, for a vast range of parameters, less than 1—in the present units this means a sweep rate of one width of the resonance curve per time span of one run-down time (but close to the high end of the hysteresis this figure falls rapidly—see Fig. 6). In conventional units this corresponds

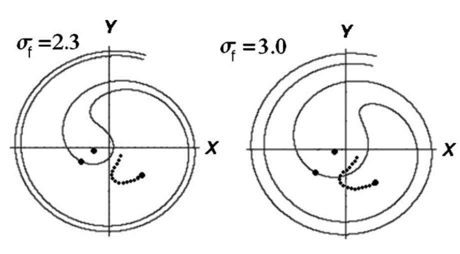

FIG. 7. Examples at $F=2.8F_{cr}$ for two values of σ_f. The value of σ_i was -2. Solid lines depict separatricies at σ_f. Also note the three FPs depicted by big black dots. The locus $\{X_f(r),Y_f(r)\}$ is shown by dotted curves—each dot for a different value of r.

[1] We could not prove this to be so for any two-basin model under any sweeping function, but we expect it for a large class of two-basin models and sweeping functions.

[2] Due to continuity of dynamical systems, a system placed at a FP at σ_i will remain there immediately after the infinitely fast ramp, i.e., $\{X_f(r),Y_f(r)\}\to$FP at σ_i as $r\to\infty$. This fact can be used to show that $r_c\propto(\sigma_f-\sigma_f^l)^{-1/2}$.

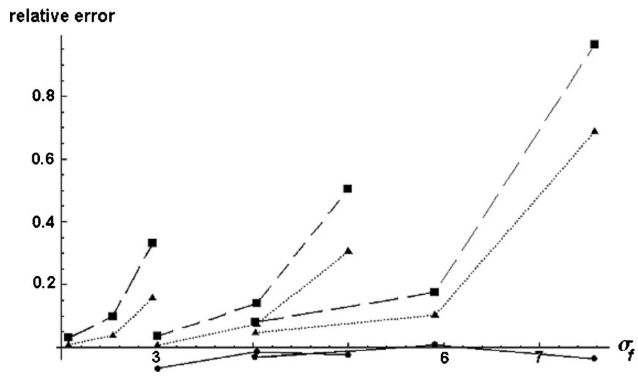

FIG. 8. Dashed lines: Relative errors between the slope of τ vs $\ln|r-r_{cr}|$ from exact Duffing and $1/(\epsilon\lambda_r)$ where λ_r is computed using the *first-order* perturbation theory in ϵ [Eqs. (2) and (3)]. $F=1.5,2,2.5$ from left to right. Solid lines: relative error between the same slope from the AE, not the full Duffing equation (only for $F=2$ and 2.5), and $1/(\epsilon\lambda_r)$; remaining errors may be due to inexactness of the linearized approximation and omission of $\mathbf{v}_r$. Dotted lines: relative error between the slope of τ vs $\ln|r-r_{cr}|$ from the exact Duffing equation and $1/(\epsilon\lambda_r)$ where λ_r is computed to second order. "Relative error" between a and b is defined as $(b-a)/a$.

to a sweep rate of $(\omega_0/Q)^2$ Hz/s. The second question is how small $|r-r_c|$ must be to induce a transient of desired duration t. From Eq. (4) we see that $|r-r_c|\approx\frac{\omega_0}{Q}(dr/dl)\times\exp[-(t\omega_0/Q)\lambda_r(\sigma_f,F)]\ \mathrm{s}^{-1}$. For any F and σ, the eigenvalue λ_r reaches maximum approximately in the middle of the hysteresis, at which point it varies approximately linearly with $F-F_{cr}$. The coefficient was found empirically to be $\approx 0.66/F_{cr}$ (recall that $F_{cr}=\frac{2^{3/2}}{3^{5/4}}$), so $\lambda_r\left(\sigma_f=\frac{\sigma_2+\sigma_1}{2},F\right)\approx 0.66(F-F_{cr})/F_{cr}$. Thus, it is easier to attain a longer transient at smaller F. The quantity dr/dl at the point of crossing the separatrix diverges at $\sigma_f=\sigma_f^l$ and becomes small (<1) for σ_f close to σ_2. This explains why the lifetime is very sensitive to $|r-r_c|$ in this region—see Eq. (4). The function dr/dl vs parameters is not fundamental. For example, it depends on the form of the ramping function, and can be computed from $\{X_f(r),Y_f(r)\}$.

A few points are in order. First, it remains to be rigorously proven (although it has been verified numerically) that the set $\{(X_f,Y_f)\}$ does not consistently cross the separatrix in some "special" way (say, always tangentially)—if it does, then our theory based on $\{(X_f,Y_f)\}$, although true, would be incomplete. This issue may be addressed in a future work by analyzing the effect of a generic perturbation of either the ramping function or the system on $\{(X_f,Y_f)\}$. This may also pave the way to understanding the generality of the phenomenon. Second, the phenomenology for the large-$\Delta\sigma$ case when the AEs (2) and (3) do not hold, yet the system is still in a weakly nonlinear regime, remains to be explored and explained.

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